The general modulus-based synchronous multi-splitting iteration method for linear complementarity problems of H_+-matrices

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Abstract—Further research is conducted on the synchronous multi-splitting iteration method to better adapt to the modern high-speed multi-processor parallel computing environments for linear complementarity problems. The general modulus-based synchronous multi-splitting iteration method (GMSM method) is proposed. The new method generalizes existing methods and has excellent parallel computational properties. The convergence of the new method is analyzed, and the convergence results are given under the different conditions. These results promote some existing related conclusions.

Keywords- linear complementarity problem; modulus-based method; multi-splitting; convergence

I. INTRODUCTION

The linear complementarity problem LCP (*q, A*) is to find a pair of vectors $z,\ r\in R^n$ so that

$$r = Az + q \geq 0, z \geq 0, z^T (Az + q) \geq 0, \quad (1)$$

where $A \in R^{n\times n}$, $q\in R^n$, and for two matrices $A=(a_{ij}),\ B=(b_{ij})\in R^{m\times n}$, $A\geq B(A > B)$ means that $a_{ij}-b_{ij}\geq 0(a_{ij}-b_{ij} > 0)$ for any i, j.

Many financial, transportation, and engineering issues ultimately boil down to the linear complementarity problem (1), such as the market supply and demand balance problem, transportation network optimization problem, free boundary problem contact problems, etc. (see [1, 2]). So far, many numerical algorithms have been achieved for the LCP (*q, A*). Especially some solvers with a special matrix A (e.g. an H-matrix and a positive definite matrix) have been pro- posed (see [3]-[11]). Recently, many researchers concentrate on the solvers transforming complementary problems into linear systems (see [3],[7]-[15]). Bai proposed the modulus-based matrix splitting and multi-splitting iteration method for LCP (q, A) (see [3] [8, 9]). Later, the modulus-based iteration method received greatly attention (see [10,11,15,16]). Zhang et al. weakened the convergence condition of the method proposed in [10], which expanded the scope of application from H- H-compatible splitting to H-splitting of the matrix A. Li and Xu generalized the modulus-based matrix splitting iteration method in two different ways, respectively, which makes the method be used in more general cases (see [11, 15]).

In this paper, a general modulus-based synchronous multi-splitting iteration method based on the method in [11] is proposed to better adapt to modern high-speed multi-processor parallel computing environments. The new method generalizes the methods in [8] and [11] and extends their application scope to more general cases.

The rest of the paper is organized as follows. Section 2 gives some definitions, notations, lemmas and briefly review of the general modulus-based matrix splitting iteration method proposed in [11]. In Section 3, we propose the new method. The convergence conclusions of the method are also given in this section. In Section 4, we conclude the work.

II. PRELIMINARIES

Let $A=(a_{ij})$ be a square matrix, then $|A|$ is a nonnegative matrix with $|A|=(|a_{ij}|)$. A^T represents the transpose of A and D_A is the diagonal matrix of A. We denote the spectral radius of A by $\rho(A)$.

Definition 2.1([17, 18]) A matrix $A=(a_{ij})\in R^{n\times n}$ is called

(1) a Z-matrix, if $a_{ij}\leq 0$ for any $i\neq j$;

(2) a P-matrix, if all its principal minors are positive;

(3) an M-matrix, if $A^{-1}\geq 0$ and A is a Z-matrix;

(4) an H-matrix, if its comparison matrix < A > is an M-matrix, where the comparison matrix

$<A>=(\tilde{a}_{ij})$ is given by

$$\tilde{a}_{ij}=\begin{cases} |a_{ij}| & if \quad i=j \\ -|a_{ij}| & if \quad i\neq j \end{cases}$$

.

(5)H+-matrix, if A is an H-matrix with positive diagonals;

(6) a strictly diagonal dominant matrix, if

$$| a_{ii} | > \sum_{j \neq i} | a_{ij} | \text{ for any } i = 1, 2, ..., n.$$

Obviously, A is a nonsingular M-matrix ⇒ A is an H_+-matrix ⇒ A is a P-matrix. A positive definite matrix is also a P-matrix. If A is an M-matrix and B is a Z-matrix, then $A \leq B$ implies that B is an M-matrix. A matrix A is an H-matrix if and only if there is a positive diagonal matrix D such that AD is a strictly diagonal dominant matrix; a matrix A is an M-matrix if and only if for any positive diagonal matrices D_1 and D_2, $D_1 A D_2$ is also an M-matrix (see [18]). It is well known that LCP (*q, A*) in (1) has a unique solution for any vector $q \in R^n$ if and only if A is a P-matrix (see [1]). So, when A is an H_+-matrix or a positive definite matrix, the LCP (*q, A*) has a unique solution.

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III. Using the Template

If $<M> - |N|$ is an M-matrix, the splitting $A = M - N$ is called an H-splitting. If $< A > = < M > - | N |$, the splitting $A = M - N$ is called an H−compatible splitting.

Provably, if $A = M - N$ is an H-splitting, then A and M are H-matrices and

$$\rho(M^{-1}N) \leq \rho(< M >^{-1} | N |) < 1$$

.

Lemma 2.1 ([18, 19]) Let $A \in R^{n \times n}$ be an H-matrix, $D = diag(A)$ be the diagonal matrix of A and $B = D - A$, then the following results hold true.

(1) A is nonsingular; (2) $| A^{-1} | \leq < A >^{-1}$

(3) $|D|$ is nonsingular and $\rho(| D |^{-1} | B |) < 1$.

Lemma 2.2 ([20]) If B is a strictly diagonal dominant matrix, then for any matrix *C*, we have

$$\| B^{-1}C \|_\infty \leq \max_i \frac{(| C | e)_i}{(< B > e)_i},$$

where $e = (1, 1, ..., 1)^T$.

For the LCP(*q,A*) in(1) by taking $z = \Omega_1(| x | + x)$, $r = \Omega_2(|x| - x)$ and $A\Omega_1 = M_{\Omega_1} - N_{\Omega_2}$, then x satisfies the system of fixed-point equation:

$$(\Omega_2 + M_{\Omega_1})x = N_{\Omega_1}x + (\Omega_2 - A\Omega_1) | x | - q. \qquad (2)$$

Here $\Omega_i \in R^{n \times n}$, $i = 1, 2$ are positive diagonal matrices. B general modulus-based matrix splitting iteration method as follows (see [11]):

Method 2.1. The General Modulus-Based Matrix Splitting Method for LCP(q, A)

Step 1. *Given an initial vector* $x^{(0)} \in R^n$ *and set* $k := 0$;

Step 2. *Compute* $x^{(k+1)} \in R^n$ *by solving the linear system*
$$(\Omega_2 + M_{\Omega_1})x^{(k+1)} = N_{\Omega_1}x^{(k)} + (\Omega_2 - A\Omega_1) | x^{(k)} | - q;$$

Step 3. *Set* $z^{(k+1)} = \Omega_1(| x^{(k+1)} | + x^{(k+1)})$;

Step 4. *If* $z^{(k+1)}$ *satisfies the given stoping rule, then terminate. Otherwise set* $k := k + 1$ *and return to Step 2.*

Next, we further study Method 2.1 and consider promoting it to multi-splitting iteration.

Firstly, we give the definition of the multi-splitting for the matrix, A.

Definition 3.1 ([12]) Let l be a given positive integer with $1 \leq l \leq n$, $A = M_k - N_k$ $(k = 1, 2, ..., l)$ be splittings of the matrix $A \in R^{n \times n}$ and $E_k \in R^{n \times n}$ $(k = 1, 2, ..., l)$ be nonnegative diagonal matrices satisfying $\sum_{k=1}^{l} E_k = I$ (the identity matrix).

Then, the collection of triples (M_k, N_k, E_k) $(k = 1, 2, ..., l)$ is called a multisplitting of the matrix *A*. In addition, the matrices E_k $(k = 1, 2, ..., l)$ are called weighting matrices.

For any given positive diagonal matrices Ω_1 and Ω_2, let $A\Omega_1 = M_k - N_k$, $(k = 1, 2, ..., l)$ be splittings of the matrix $A\Omega_1 \in R^{n \times n}$, from Method 2.1 we know that if x satisfies any one of the following equations

$$(\Omega_2 + M_k)x = N_k x + (\Omega_2 - A\Omega_1) | x | - q, \quad (k = 1, 2, ..., l) \qquad (3)$$

Then $z = \Omega_1(| x | + x), r = \Omega_2(| x | - x)$

is a solution of the $LCP(q, A)$. With the equivalent reformulations (3) and (4), we can establish the general modulus-based synchronous multi-splitting iteration method (GMSM method,

The new method has excellent parallel computational properties. Furthermore, an appropriate choice of the matrices M_k and E_k will balance the tasks carried by each processor in a multiprocessing system.

Method 3.1. The GMSM Iteration Method for LCP(*q*, *A*)

Step 1. *Given an initial vector* $x^{(0)} \in R^n$ *and set* $m := 0$;

Step 2. *For* $k = 1, 2, ..., l$, *compute* $x^{(m,k)} \in R^n$ *by solving the linear system*
$$(\Omega_2 + M_k)x^{(m,k)} = N_k x^{(m)} + (\Omega_2 - A\Omega_1) | x^{(m)} | - q$$

Step 3. *Let* $x^{(m+1)} = \sum_{k=1}^{l} E_k x^{(m,k)}$, *and* $z^{(m+1)} = \Omega_1(|x^{(m+1)}| + x^{(m+1)})$;

Step 4. *If* $z^{(m+1)}$ *satisfies the given stoping rule, then terminate. Otherwise set* $m := m + 1$ *and return to Step 2.*

Theorem 3.1 Let $A \in R^{n\times n}$ be an H+-matrix. Both Ω_1 and Ω_2 are positive diagonal matrices. (M_k, N_k, E_k) $(k = 1,2,...,l)$ is a multi-splitting of the matrix $A\Omega_1$, and $A\Omega_1 = M_k - N_k$, $(k = 1,2,...,l)$ are all H-splittings. Then the iteration sequence $\{z^{(m)}\}_{m=1}^{+\infty}$ generated by Method 3.1 converges to the unique solution z^* of the LCP(q, A) for any initial vector $x^{(0)} \in R^n$ provided.

$$\Omega_2 e \ > \ D_A\Omega_1 e - D_k^{-1}(< \ M_k > - | N_k |)D_k e,$$

for any positive diagonal matrix D_k such that $(< M_k > -|N_k|)D_k$ $(k = 1,2,...,l)$ is a strictly diagonal dominant matrix.

Proof. As $<A>$ is an M-matrix, and Ω_1 is a positive diagonal matrix, it follows that $<A>\ \Omega_1$ is an M-matrix, and further $A\Omega_1$ is an H+-matrix. From $A\Omega_1 = M_k - N_k$ are the H-splittings of the H+-matrix $A\Omega_1$, we can conclude that M_k and $\Omega_2 + M_k$ are both H-matrices. Thus, by Lemma 2.1, $|(\Omega_2 + M_k)^{-1}| \le (\Omega_2 + < M_k >)^{-1}$ holds true.

From Method 3.1, we have

$$x^{(m,k)} = (\Omega_2 + M_k)^{-1}[N_k x^{(m)} + (\Omega_2 - A\Omega_1)\,|\,x^{(m)}| - q], k = 1,2,...,l. \quad (5)$$

So

$$x^{(m+1)} = \sum_{k=1}^{l} E_k x^{(m,k)} = \sum_{k=1}^{l} E_k(\Omega_2 + M_k)^{-1}[N_k x^{(m)} + (\Omega_2 - A\Omega_1)\,|\,x^{(m)}| - q]. \quad (6)$$

Let

$$x^* = \sum_{k=1}^{l} E_k(\Omega_2 + M_k)^{-1}[N_k x^* + (\Omega_2 - A\Omega_1)\,|\,x^*| - q],$$

Then

$$|\,x^{(m+1)} - x^*| \le \sum_{k=1}^{l} E_k\,|(\Omega_2 + M_k)^{-1}|\,[|\,N_k\,||\,x^{(m)} - x^*| + |\,\Omega_2 - A\Omega_1\,||\,x^{(m)} - x^*|]$$

$$\le \sum_{k=1}^{l} E_k\,|(\Omega_2 + < M_k >)^{-1}|\,[|\,N_k\,| + |\,\Omega_2 - A\Omega_1\,|]\,|\,x^{(m)} - x^*|$$

Denote

$T_k = (\Omega_2 + < M_k >)^{-1}[|\,N_k\,| + |\,\Omega_2 - A\Omega_1\,|]$ and $L_{GMSM} = \sum_{k=1}^{l} E_k T_k$.

Let $A_{\Omega_1}^{(k)} =< M_k > - |\,N_k\,|$, then $A_{\Omega_1}^{(k)}$ is an M-matrix and $< M_k >=|\,N_k\,| + A_{\Omega_1}^{(k)}$. It is obtained that $A_{\Omega_1}^{(k)} \le <A>\ \Omega_1$ from $< M_k > - |\,N_k\,| \le < M_k - N_k >$. As $A_{\Omega_1}^{(k)}$ is an H-matrix, there is a positive diagonal matrix D_k such that $A_{\Omega_1}^{(k)} D_k$ is a strictly diagonal dominant matrix. Since $(\Omega_2 + < M_k >)D_k \ge A_{\Omega_1}^{(k)} D_k$, $(\Omega_2 + < M_k >)D_k$ is also a strictly diagonal dominant matrix. Therefore, from Lemma 2.2, we have

$$\| D_k^{-1}T_k D_k \|_\infty = \|(\Omega_2 D_k + < M_k > D_k)^{-1}(|\,N_k\,|\,D_k + |\,\Omega_2 - A\Omega_1\,|\,D_k)\|_\infty$$
$$\le \max_i \frac{(|\,N_k\,|\,D_k e + |\,\Omega_2 - A\Omega_1\,|\,D_k e)_i}{[(\Omega_2 + < M_k >)D_k e]_i}$$
$$= \max_i \frac{(|\,N_k\,|\,D_k e)_i + |\,\omega_i' - a_{ii}\omega_i\,|\,d_i^{(k)} + \sum_{j\ne i}|\,a_{ij}\,|\,\omega_j d_j^{(k)}}{[(\Omega_2 + |\,N_k\,| + A_{\Omega_1}^{(k)})D_k e]_i} \quad (7)$$

where

$$\Omega_1 = diag(\omega_1, \omega_2, ..., \omega_n), \Omega_2 = diag(\omega_1', \omega_2', ..., \omega_n'), D_k = diag(d_1^{(k)}, d_2^{(k)}, ..., d_n^{(k)}).$$

Case I. If $\Omega_2 e \ge D_A\Omega_1 e$, that is $\omega_i' \ge a_{ii}\omega_i$, then

$$(|\,N_k\,|\,D_k e)_i + |\,\omega_i' - a_{ii}\omega_i\,|\,d_i^{(k)} + \sum_{j\ne i}|\,a_{ij}\,|\,\omega_j d_j^{(k)}$$
$$= (|\,N_k\,|\,D_k e)_i + [(\Omega_2 - < A > \Omega_1)D_k)e]_i$$
$$\le [(\Omega_2 + |\,N_k\,| - A_{\Omega_1}^{(k)})D_k)e]_i$$

Thus,

$$\| D_k^{-1}T_k D_k \|_\infty \le \max_i \frac{[(\Omega_2 + |\,N_k\,| - A_{\Omega_1}^{(k)})D_k e]_i}{[(\Omega_2 + |\,N_k\,| + A_{\Omega_1}^{(k)})D_k e]_i} < 1.$$

Case II. $D_A\Omega_1 e - D_k^{-1}(< M_k > - |\,N_k\,|)D_k e < \Omega_2 e < D_A\Omega_1 e$,

that is

$$[(D_A\Omega_1 - D_k^{-1}(< M_k > - |\,N_k\,|)D_k)e]_i < \omega_i' < a_{ii}\omega_i,$$

then

$$(|\,N_k\,|\,D_k e + |\,\Omega_2 - A\Omega_1\,|\,D_k e)_i$$

$$\| D_k^{-1}T_k D_k \|_\infty = \|(\Omega_2 D_k + < M_k > D_k)^{-1}(|\,N_k\,|\,D_k + |\,\Omega_2 - A\Omega_1\,|\,D_k)\|_\infty$$
$$\le \max_i \frac{(|\,N_k\,|\,D_k e + |\,\Omega_2 - A\Omega_1\,|\,D_k e)_i}{[(\Omega_2 + < M_k >)D_k e]_i}$$
$$= \max_i \frac{(|\,N_k\,|\,D_k e)_i + |\,\omega_i' - a_{ii}\omega_i\,|\,d_i^{(k)} + \sum_{j\ne i}|\,a_{ij}\,|\,\omega_j d_j^{(k)}}{[(\Omega_2 + |\,N_k\,| + A_{\Omega_1}^{(k)})D_k e]_i} \quad (8)$$

From (8), the assumption $\Omega_2 e < D_A\Omega_1 e$ and $A_{\Omega_1}^{(k)} D_k$ is a strictly diagonal dominant matrix, we have

$$\| D_k^{-1}T_kD_k \|_\infty = \| (\Omega_2 D_k + < M_k > D_k)^{-1}(| N_k | D_k + | \Omega_2 - A\Omega_1 | D_k) \|_\infty$$
$$\le \max_i \frac{(| N_k | D_k e + | \Omega_2 - A\Omega_1 | D_k e)_i}{[(\Omega_2 + < M_k >)D_k e]_i}$$
$$= \max_i \frac{(| N_k | D_k e)_i + | \omega_i' - a_{ii}\omega_i | d_i^{(k)} + \sum_{j\ne i} | a_{ij} | \omega_j d_j^{(k)}}{[(\Omega_2 + | N_k | + A_{\Omega_1}^{(k)})D_k e]_i}$$

from (7), which implies $\| D_k^{-1}T_kD_k \|_\infty < 1$. Therefore,

$$\rho(T_k) = \rho(D_k{}^{-1}T_kD_k) \le \| D_k{}^{-1}T_kD_k \|_\infty < 1.$$

It is followed that

$$\rho(L_{GMSM}) = \rho(\sum_{k=1}^{l} E_k D_k^{-1} T_k D_k)$$
$$\le \max_{1\le i\le n} \sum_{k=1}^{l} e_i^{(k)} \| D_k^{-1}T_kD_k \|_\infty$$
$$< 1$$

This implies Method 3.1 converges for any initial vector $x^{(0)} \in R^n$.

Remark: 1. The different selections of the diagonal matrices Ω_1 and Ω_2 would make the solving

methods more choices and more efficient than those proposed before. Some existing methods are special cases of our method. For example, in Method 3.1, if $l = 1$, we would obtain the general modulus-based iteration method proposed in [11]. And if $\Omega_1 = \gamma I$, $\Omega_2 = \Omega$, $l = 1$, we would obtain the modulus-based matrix splitting iteration methods in [3].

2. By solving a linear system $(< M_k > - | N_k |)x = p$, we can obtain the positive diagonal matrix $D_k = diag(x)$ in Theorem 3.1, here p is an any positive vector. Let $p = e$, then $D_k = diag((< M_k > - | N_k |)^{-1}e)$. So the conditions of the Theorem 3.1 can be simplified as $\Omega_2 e > (D_A\Omega_1 - D^{-1})e$. Especially when $l = 1$, this conclusion is similar to that in [13, 14].

3. Let $\Omega_1 = \gamma^{-1}I$ and $\Omega_2 = \gamma^{-1}\Omega$, then Method 3.1 would reduce to the method in [8], in where the convergence results were obtained when $A = M_k - N_k, (k = 1, 2, ..., l)$ are H-compatible splittings. It should be noted that an H-compatible splitting of an H-matrix is also an H-splitting, but not vice versa. For example, let

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, M = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}, N = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

It is easy to verify that A is an H-matrix, *and* $A = M - N$ is an H-splitting instead of an H-compatible splitting. Thus compared to the method in [8], our method has a wider range of applications but weaker conditions.

In fact, for the H-compatible splittings of $A\Omega_1$, we can easily obtain the following result.

Corollary 3.2 Let A be an H_+-matrix. $\Omega_2 = diag(\omega_1', \omega_2', \cdots, \omega_n')$ is a positive diagonal matrix. For the integer $k = 1, 2, ..., l (1 \le l \le n)$, (M_k, N_k, E_k) is a multisplitting of the matrix $A\Omega_1$, and $A\Omega_1 = M_k - N_k$ are all H-compatible splittings. Then, the iteration sequence $\{z^{(m)}\}_{m=1}^{+\infty}$ generated by Method 3.1 converges to the unique solution z^* of the LCP (q, A) for any initial vector $x^{(0)} \in R^n$ provided

$$\Omega_2 e > (D_A - < A >)\Omega_1 e,$$

that is
$$\omega_i' > \sum_{j\ne i} | a_{ij} | \omega_j,$$

where $\Omega_1 = diag(\omega_1, \omega_2, \cdots, \omega_n)$ is a positive diagonal matrix such that $< A > \Omega_1$ is a strictly diagonal dominant matrix.

An M-splitting of an M-matrix is an H-compatible splitting and also an H-splitting. Therefore, the following result holds.

Corollary 3.3 Let A be a strictly diagonal dominant M-matrix. Both $\Omega_1 = diag(\omega_1, \omega_2, \cdots, \omega_n)$ and $\Omega_2 = diag(\omega_1', \omega_2', \cdots, \omega_n')$ are positive diagonal matrices. $(M_k, N_k, E_k)(k = 1, 2, ..., l)$ is a multi-splitting of the matrix $A\Omega_1$, and $A\Omega_1 = M_k - N_k$, $(k = 1, 2, ..., l)$ are all M-splittings. Then, the iteration sequence $\{z^{(m)}\}_{m=1}^{+\infty}$ generated by Method 3.1 converges to the unique solution z^* of the $LCP(q, A)$ for any initial vector $x^{(0)} \in R^n$ provided $\omega_i' > \sum_{j\ne i} | a_{ij} | \omega_j$.

1. Conclusion

The paper proposes the GMSM method for LCP(q, A). We analyzed the convergence of the method and gave the convergence results under particular conditions. Due to the selection of Ω_1, Ω_2 and the expansion of the matrix splitting range, the method and the corresponding convergence results generalized the related conclusions. If the special splitting of the matrix $A\Omega_1$ are chosen, the method proposed will yield a series of methods, such as the GMSM SOR method and the GMSM AOR method, etc, which will be the issues we need to study next.

Acknowledgements

The authors would like to appreciate the suggestions from referee. This research was funded by the National Natural Science Foundation of China (grant number: 12271431).

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